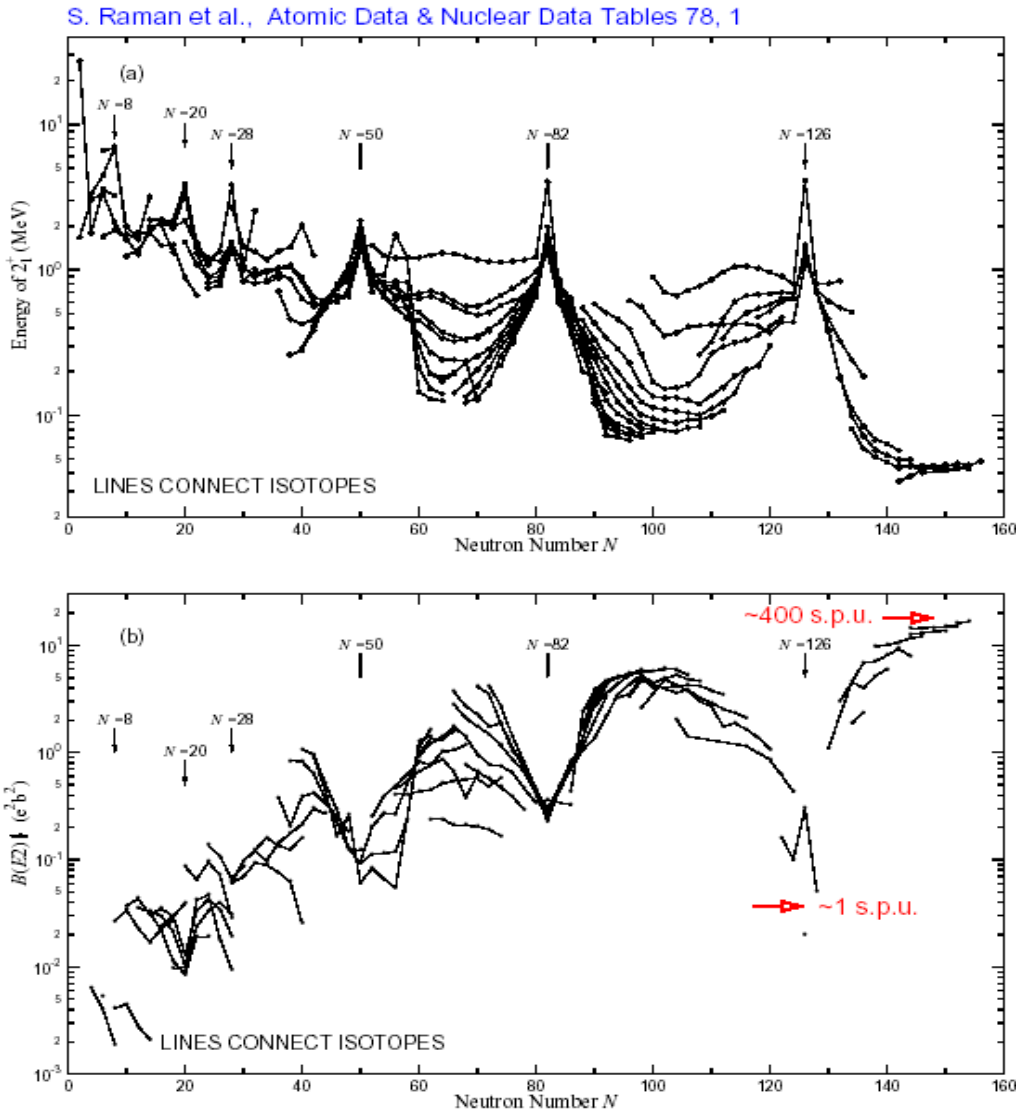
Nuclear Structure from Gamma-Ray Spectroscopy

2019 Postgraduate Lectures

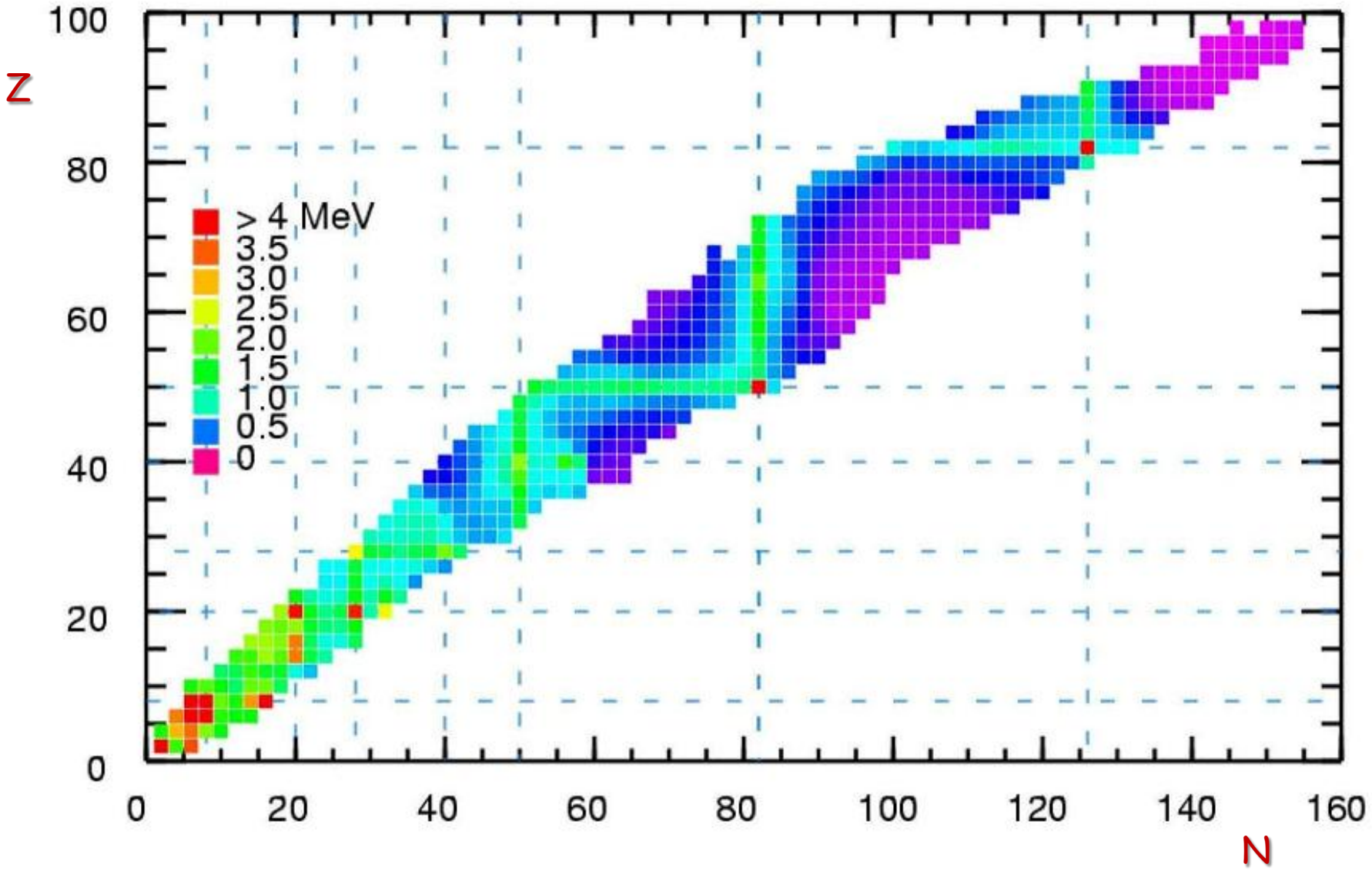
Lecture 3: Spherical & Deformed Shell Model

Experimental Shell Effects

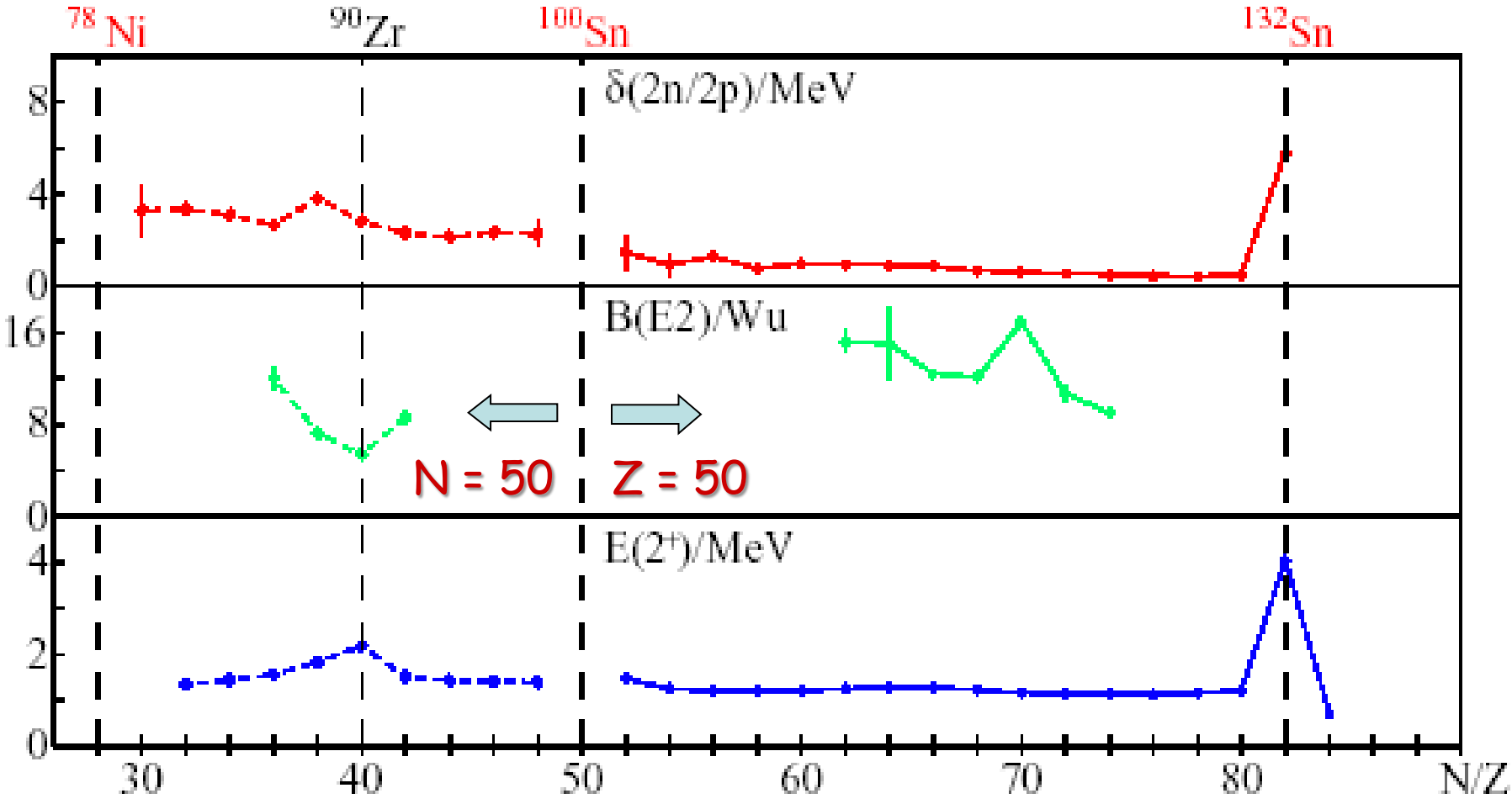


- The energies of the first excited 2^+ states in nuclei peak at the magic numbers of protons or neutrons
- 'B(E2)' values ($\propto 1/\tau$ where τ is the mean lifetime) of the 2^+ states reach a minimum at the magic numbers
- 'Magic' nuclei are spherical and the least collective

First 2^+ Energies

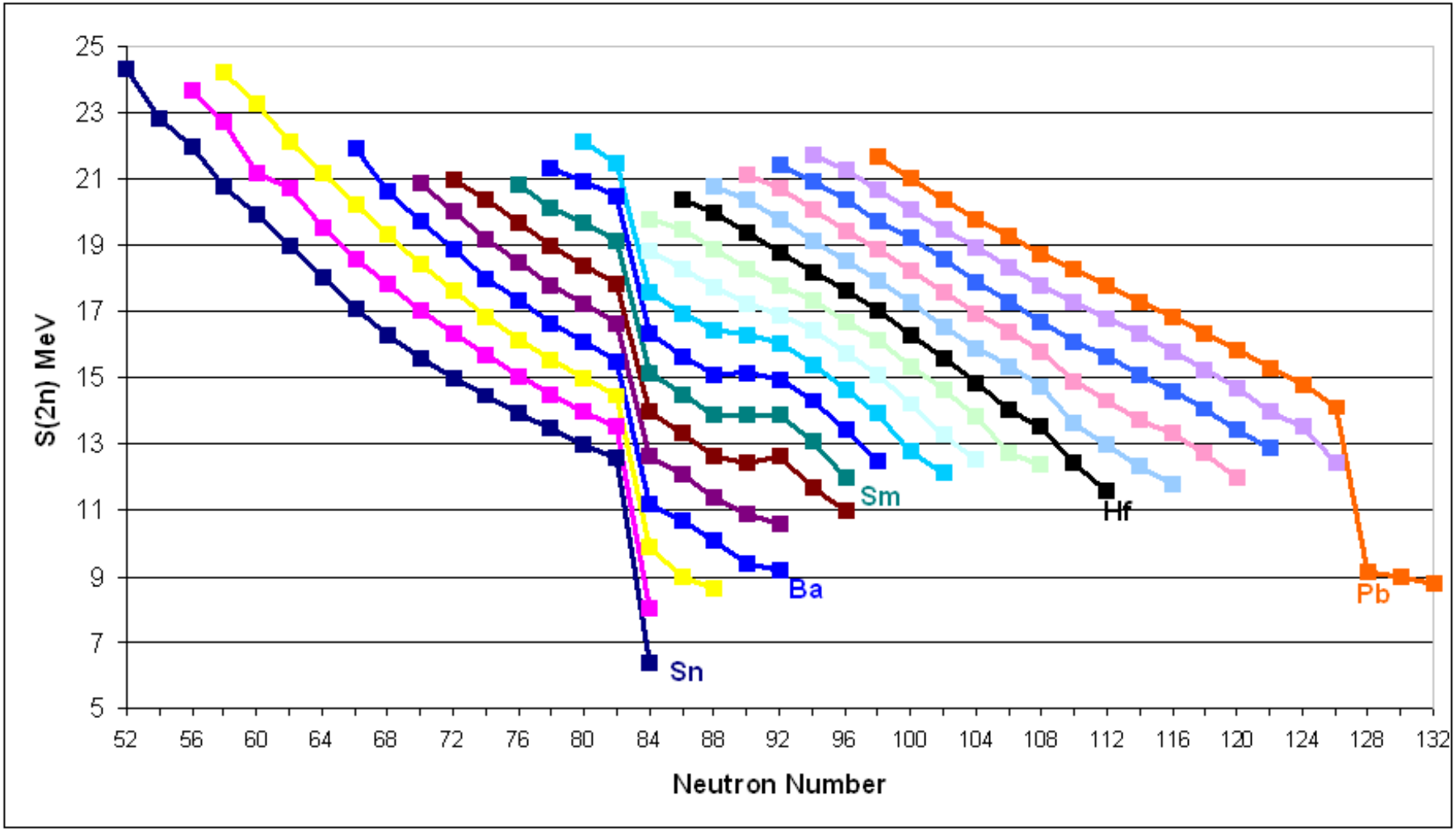


Systematics Near Z(N) = 50



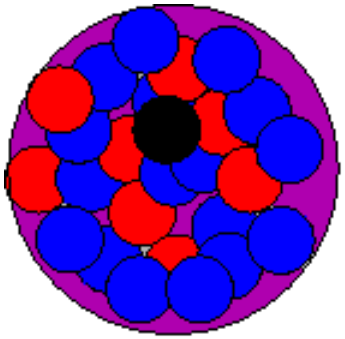
- ^{100}Sn (Z=N=50) and ^{132}Sn (N=82) are doubly magic nuclei

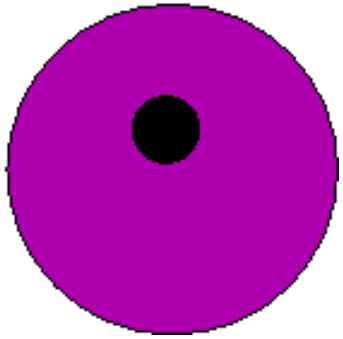
Neutron Separation Energies



Shell Model – Mean Field

N nucleons in a nucleus





A nucleon in the Mean Field of N-1 nucleons

- Assumption – ignore detailed two-body interactions
- Each particle moves in a state independent of other particles
- The Mean Field is the average smoothed-out interaction with all the other particles
- An individual nucleon only experiences a central force

Shell Model Hamiltonian

- If the short range interaction potential between two nucleons i and j is $v(r_{ij})$, then the average potential acting on each particle is:

$$V_i(r_i) = \langle \Sigma_j v(r_{ij}) \rangle$$

- The Hamiltonian, $H = \Sigma_i T_i + \Sigma_{ij} v(r_{ij})$, can be rewritten:

$$H' = \Sigma_i[T_i + V_i(r_i)] \quad + \quad \lambda\,[\Sigma_{ij} v(r_{ij}) - \Sigma_i V_i(r_i)]$$

mean field — residual interaction

- For $\lambda = 1$, $H' = H$. The shell model assumption is that $\lambda \rightarrow 0$, i.e. the central interaction is much larger than the residual interactions

Choice of Potential

- A central potential $V(r_i)$ only depends on the distance r_i and is made up of a superposition of short-range internucleonic potentials:

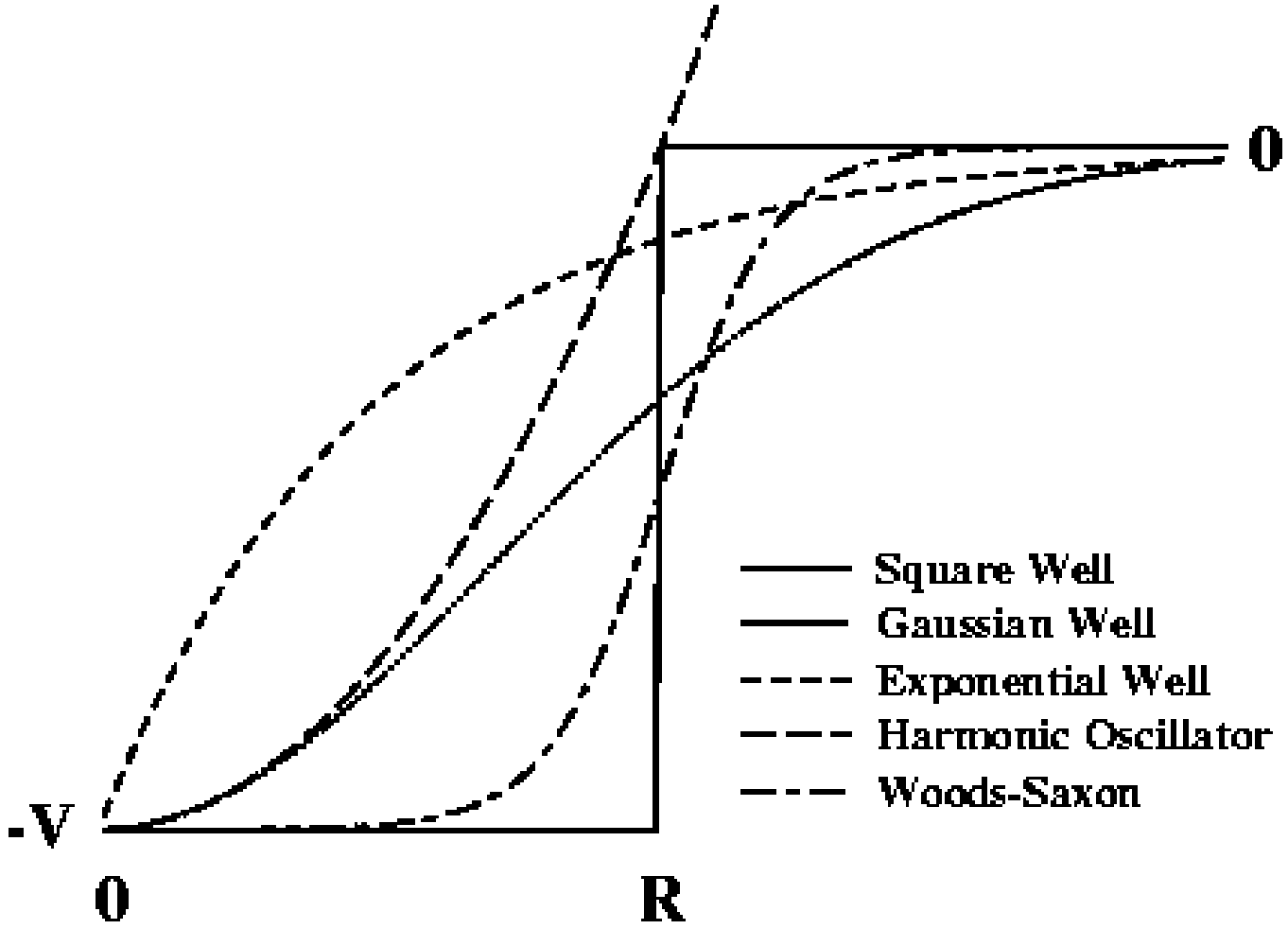
$$V(r_i) = \int v|r_i - r'|\, \rho(r')\, dr'$$

- '$\rho(r')$' is the density distribution of the nucleus
- The internucleonic potential may be represented by a delta function: $v(r_{ij}) = -V_0\, \delta(r_{ij})$
- Then: $V(r_i) = V_0\, \rho(r)$
- The Schrödinger equation is: $[T + V]\, \Psi(r) = E\, \Psi(r)$

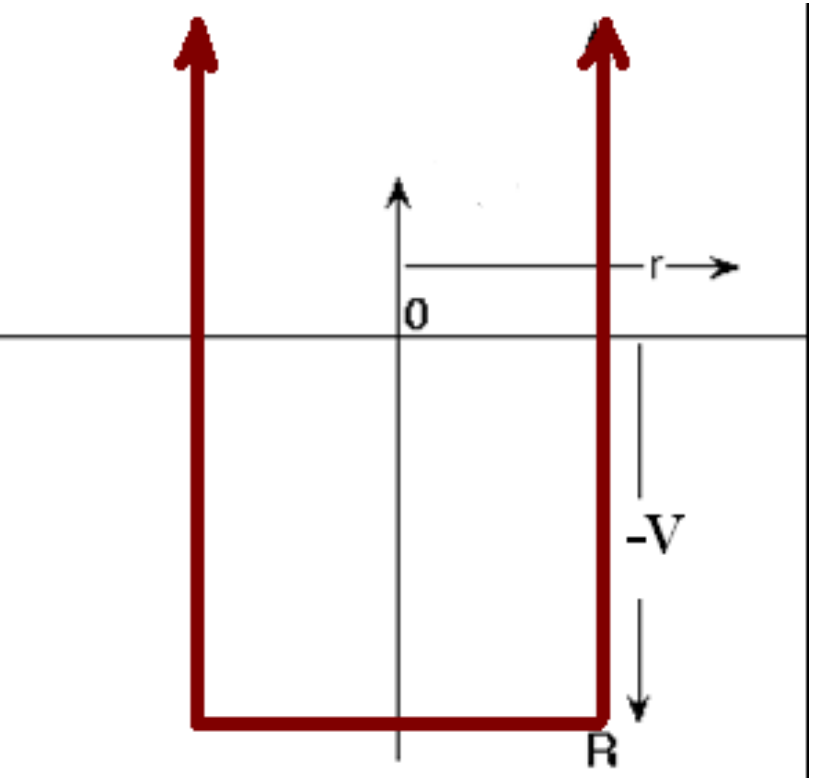
Some Potential Wells

- Square Well: $V(r) = -V_0$ for $r \leq R_0$
  $= 0$ for $r > R_0$
- Gaussian Well: $V(r) = -V_0 \exp[-(r/a)^2]$
- Exponential Well: $V(r) = -V_0 \exp[-2r/a]$
- Yukawa Well: $V(r) = -(V_0/r) \exp[-r/a]$
- Harmonic Oscillator: $V(r) = -V_0[1-(r/R_0)^2]$
- Woods-Saxon: $V(r) = -V_0 / \{1 + \exp[(r-R_0)/a]\}$

Well Comparisons



Square Well Potential



Infinite square well potential

- Simplest form of potential
- Since we have a spherically symmetric potential we can separate the solutions into angular and radial parts
- Radial solutions are Bessel functions which satisfy the boundary condition $R_{n\ell}(R) = 0$
- The eigenvalues are:

$$R_{n\ell} = \{A/\sqrt{(\kappa r)}\}\, J_{\ell+\frac{1}{2}}(\kappa r)$$

where A is a constant and κ is the wave number of the nucleon:

$$\kappa^2 = (2M/\hbar^2)[E_{n\ell} + V]$$

Square Well Labels

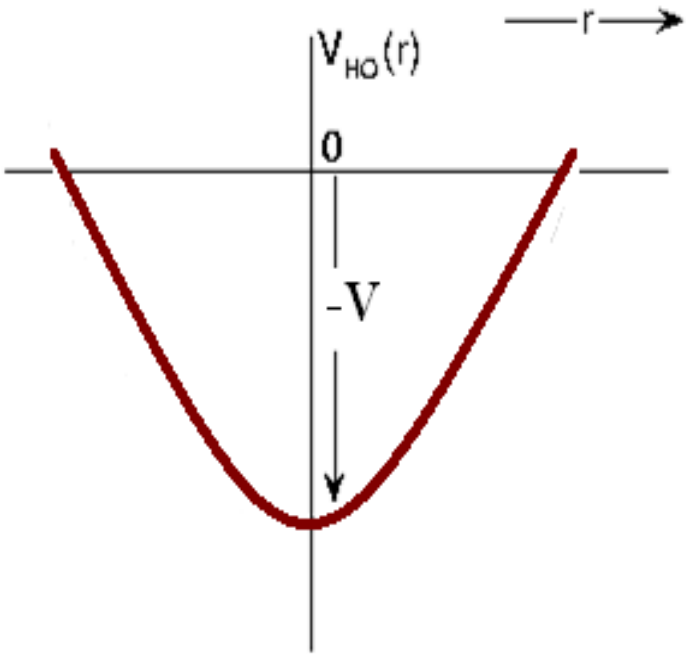
- The levels are labelled by n and ℓ ('s' = 0, 'p' = 1, 'd' = 2, 'f' = 3, 'g' = 4, 'h' = 5, 'i' = 6, 'j' = 7, 'k' = 8)

- Each level has $2(2\ell + 1)$ substates

- The first few levels (different from H atom):

Level	Occupation	Total
1s	2	2
1p	6	8
1d	10	18
2s	2	20
1f	14	34
2p	6	40

Harmonic Oscillator Potential



Simple harmonic oscillator potential

- Easy to handle analytically
- Form of potential:

$$V_{HO}(r) = -V + \tfrac{1}{2}mr^2\omega^2$$

- Solutions are Laguerre polynomials
- Eigenenergies are labelled by the oscillator quantum number N:

$$E_N = (N + 3/2)\,\hbar\omega$$

- For each N there are degenerate levels with n and ℓ that satisfy:

$$2(n-1) + \ell = N, \quad N \geq 0, \quad 0 \leq \ell \leq N$$

- The parity of each shell is $(-1)^N$

Harmonic Oscillator Degeneracies

- For each N there are degenerate energy levels with n and ℓ that satisfy:

$$2(n-1) + \ell = N, \quad N \geq 0, \quad 0 \leq \ell \leq N$$

- Even N contains only ℓ even states; odd N, odd ℓ

- The degeneracy condition is:

$$\Delta\ell = 2 \text{ and } \Delta n = 1$$ (e.g. N = 4 3s, 2d, 1g orbits)

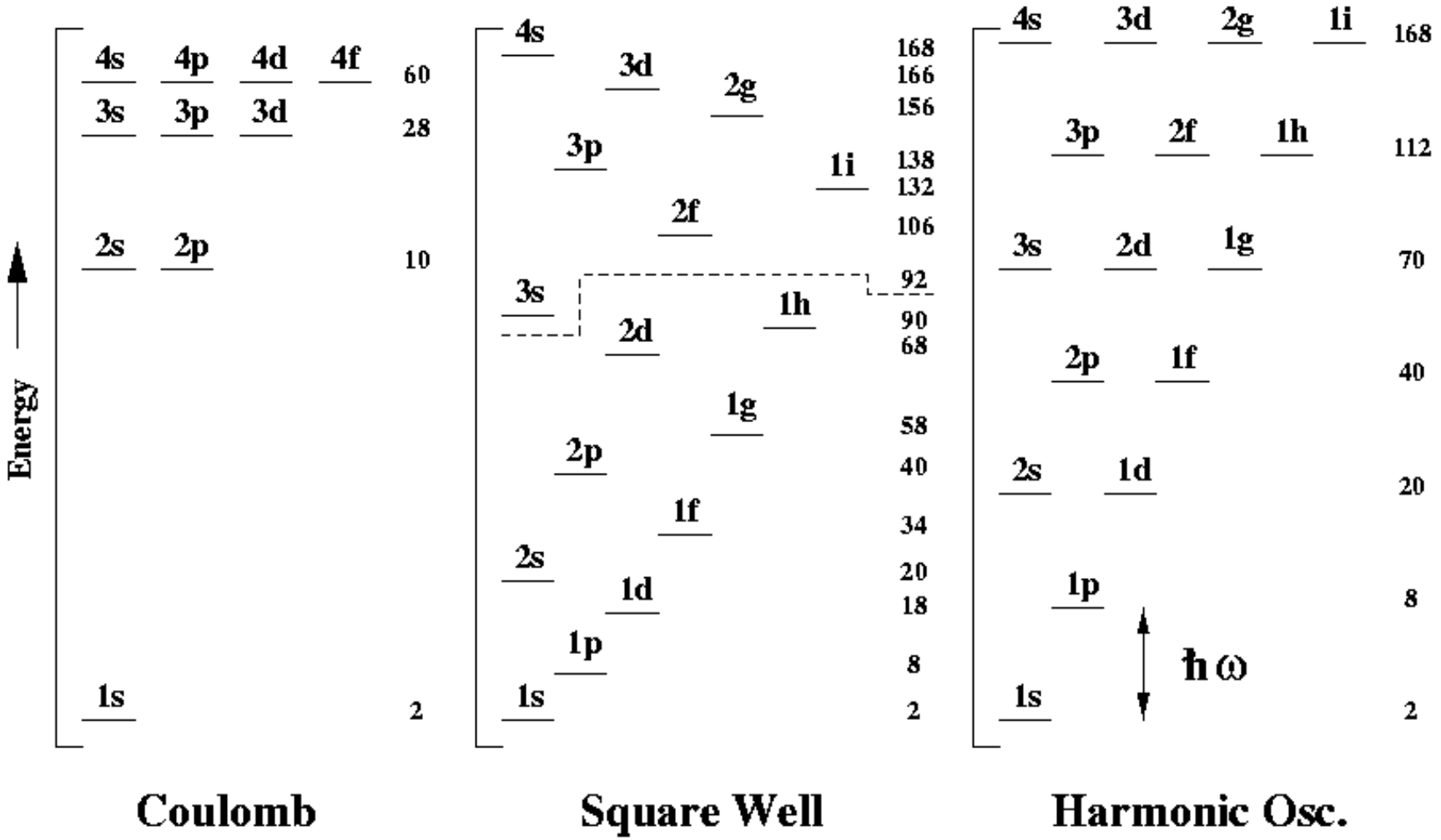
- It is the fundamental reason for shell structure, i.e. clustering of levels

Harmonic Oscillator Labels

- The number of degenerate levels for a given N is (N+1)(N+2)

N	allowed ℓ	E_N	Occupation	Total
0	0	3/2	2	2
1	1	5/2	6	8
2	2,0	7/2	12	20
3	3,1	9/2	20	40
4	4,2,0	11/2	30	70
5	5,3,1	13/2	42	112

(Wrong) Magic Numbers



Spin-Orbit Potential

- In Atomic Physics the spin-orbit interaction comes about due to the interaction of an electron's magnetic moment with the magnetic field generated by its motion about the nucleus

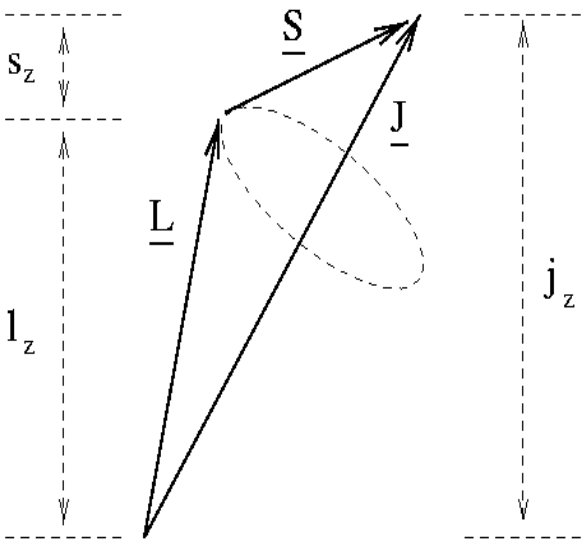
- A similar interaction was introduced for nuclei to empirically fit the observed magic numbers

- A term is added to the potential:

$$V(r) \rightarrow V(r) + \mu \underline{\ell}.\underline{s}$$

- The new term makes the force felt by a nucleon dependent on the direction of its spin

Spin Orbit Energy



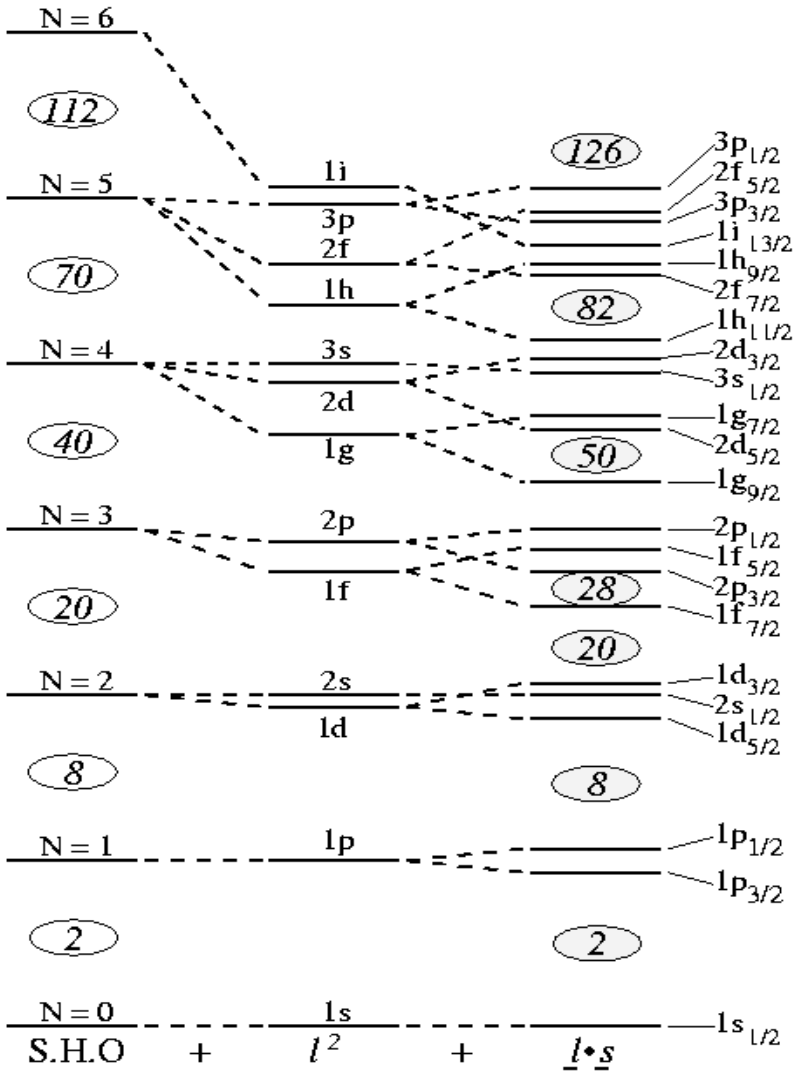
The vectors L and S precess about J

- The spin-orbit term does not violate spherical symmetry and leaves ℓ, j and j_z as good quantum numbers, although ℓ_z and s_z are not
- The spin-orbit energy is:

$$E_{\ell.s} = \{[4j(j+1)-4\ell(\ell+1)-1]/8\}\hbar^2\mu$$

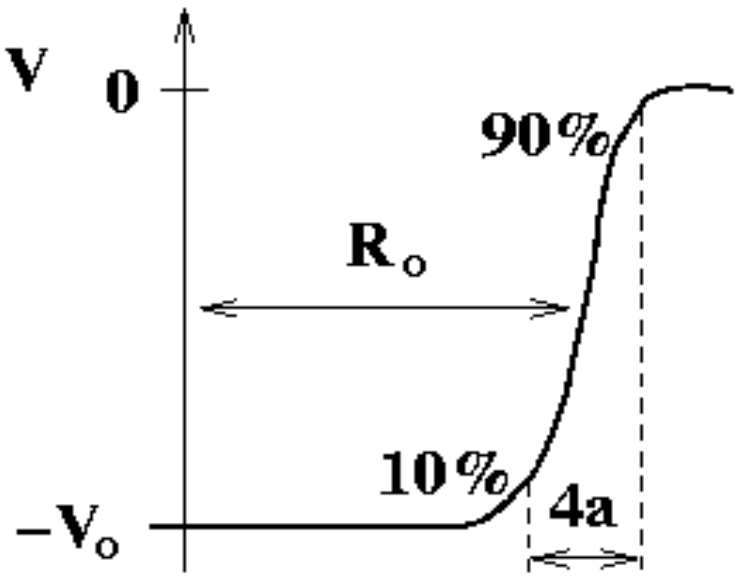
- By making $\mu < 0$, the magic numbers can be reproduced
- States with $j = \ell + \frac{1}{2}$ are lower in energy than those states with $j = \ell - \frac{1}{2}$ (opposite way round to spin-orbit interaction in atoms !)

Modified Harmonic Oscillator



- The harmonic oscillator shells are shown to the left in this diagram
- In the middle, an ℓ^2 term is added to make the potential more realistic
- A spin orbit term $\underline{\ell}.\underline{s}$ is added to the right with its strength adjusted to obtain the correct nuclear magic numbers

Woods-Saxon Potential



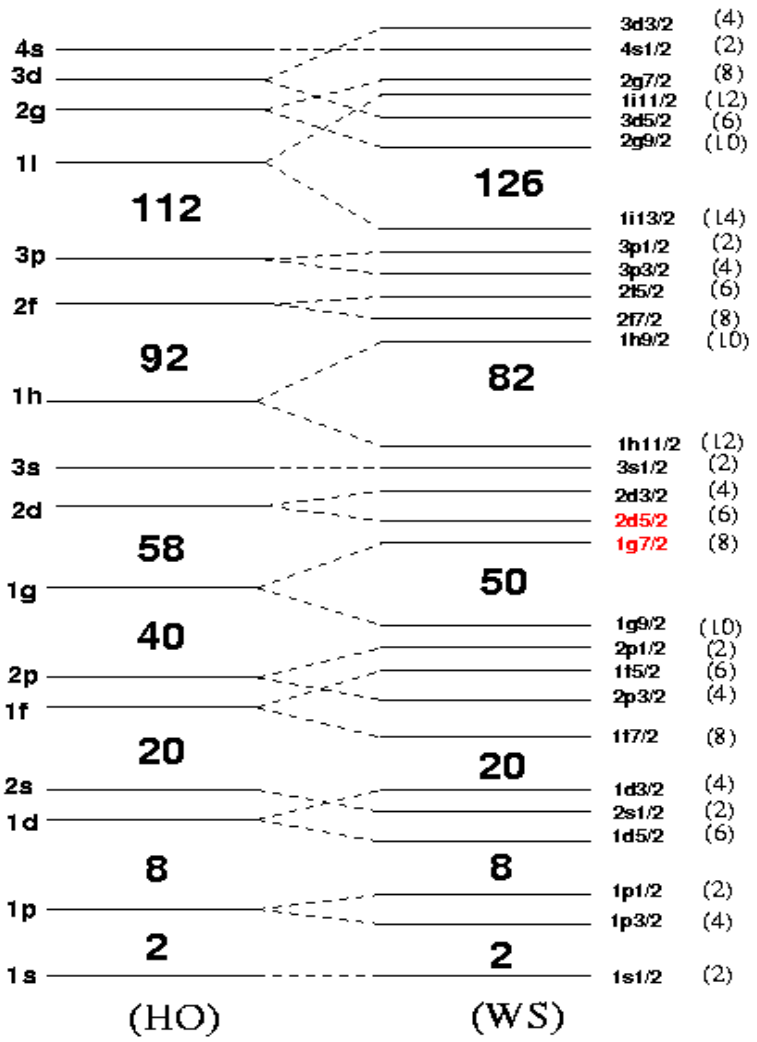
V_0 ~ 50 MeV
R_0 ~ 6-7 fm (A=125-190)
a ~ 0.5 fm
4a = 'skin thickness'

- The Woods-Saxon (WS) nuclear potential is 'supposedly' the most realistic

- The potential has the form:

$$V(r) = -V_0 / \{ 1 + \exp[(r - R_0) / a] \}$$

WS vs. MHO Potentials



- The Woods-Saxon (WS) potential is the most realistic
- The ℓ^2 term in the Modified Harmonic Oscillator (MHO) potential flattens the bottom of the potential making it look more like the Woods-Saxon shape
- There are slight differences between the MHO and WS energy levels, e.g. the ordering of the $2d_{5/2}$ and $1g_{7/2}$ levels is interchanged

Anisotropic Harmonic Oscillator

- The Anisotropic Harmonic Oscillator (AHO) potential for a spheroidal nucleus deformed along the z-axis may be written:

$$V_{osc} = \tfrac{1}{2}M[\omega_{\perp}^{2}(x^2 + y^2) + \omega_z^2 z^2]$$

- Here $\omega_{\perp}$ and ω_z represent the frequencies of the simple harmonic motion perpendicular and parallel to the nuclear symmetry axis, respectively, and are functions of the nuclear deformation:

$$\omega_z \approx \omega_0[1 - 2/3\,\delta], \quad \omega_{\perp} \approx \omega_0[1 + 1/3\,\delta]$$

and $\omega_0^3 = \omega_{\perp}^2\omega_z$ for volume conservation

Harmonic Oscillator Quantum

- The Harmonic Oscillator quantum ω_0 is usually taken to have an isospin dependence:

$$\hbar\omega_0 = 41\, A^{-1/3}\,[1 \pm (N-Z)/3A] \quad \text{MeV}$$

where the minus sign is used for protons and the plus sign for neutrons

- In the 'stretched' coordinate system, the potential may then be written simply as:

$$V_{osc} = \tfrac{1}{2}\hbar\omega_0(\varepsilon_2)\,\rho^2\,[1 - 2/3\,\varepsilon_2\,P_2(\cos\theta_t)]$$

where ε_2 is (yet) another deformation parameter

δ, β and ε Parameters

- Three deformation parameters are often used:
  1. Delta: $\delta = \Delta R/R$
  2. Epsilon: ε_2 defines a rotational ellipsoid
  3. Beta: β_2 defines a rotational quadrupoloid

- If the deformation is not so large, then the following approximations hold:

  $\varepsilon_2 \approx 0.946\ \beta_2\ (1 - 0.1126\beta_2)$

  $\delta \approx 0.946\ \beta_2\ (1 - 0.2700\beta_2)$

- Also the hexadecapole β_4 parameter has <u>opposite</u> sign to the ε_4 parameter: $\varepsilon_4 \approx -0.85\ \beta_4$

Solutions of the AHO

- The eigenvalues of the AHO potential are:

  $E(n_z,n_\perp) = [n_z + \frac{1}{2}]\,\hbar\omega_z + [n_\perp + 1]\,\hbar\omega_\perp$

  or

  $E(N,n_z,n_\perp) \approx [N + 3/2]\,\hbar\omega_0 - 1/3\delta[2n_z - n_\perp]\,\hbar\omega_0$

  with $N = n_z + n_\perp$

- The latter expression is simply the energies of a Spherical Harmonic Oscillator minus a correction term, proportional to the deformation

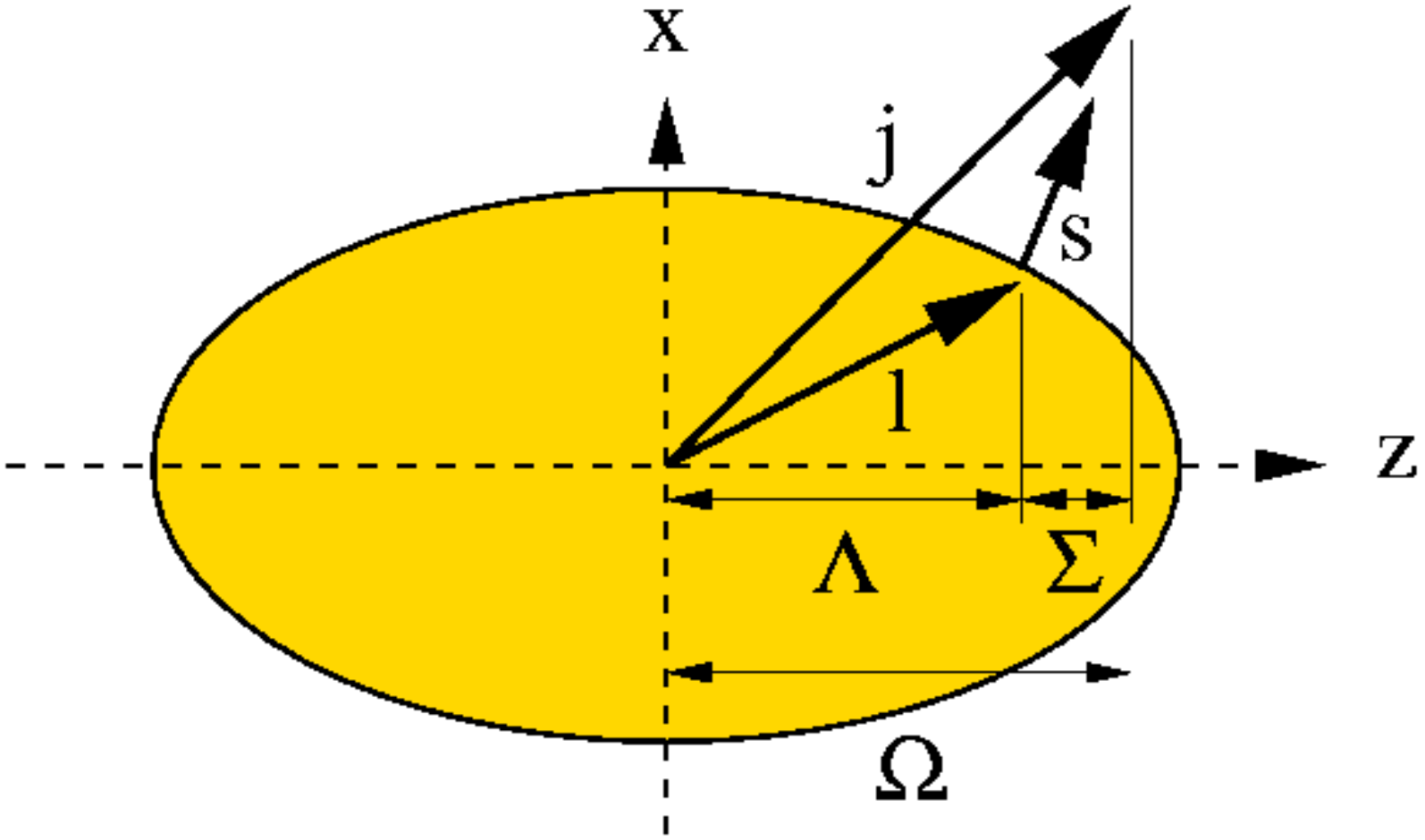
- The energy levels are labelled by the asymptotic quantum numbers:

$$\Omega^\pi\,[N\ n_z\ \Lambda]$$

AHO Labels

- The energy levels are labelled by the asymptotic quantum numbers: $\Omega^{\pi}\ [N\ n_z\ \Lambda]$
- 'N': $N = n_x + n_y + n_z$ $(= n_z + n_\perp)$ is the oscillator quantum number
- 'n_z': n_z describes the z-axis component of N
- 'Λ': $\Lambda = \ell_z$ is the projection of ℓ onto the z-axis
- 'Ω': $\Omega = \Lambda + \Sigma$ is the projection of $j = \ell + s$ onto the z-axis
- 'π': $\pi = (-1)^{\ell}$ is the parity of the state

The Λ, Σ, Ω Quantum Numbers



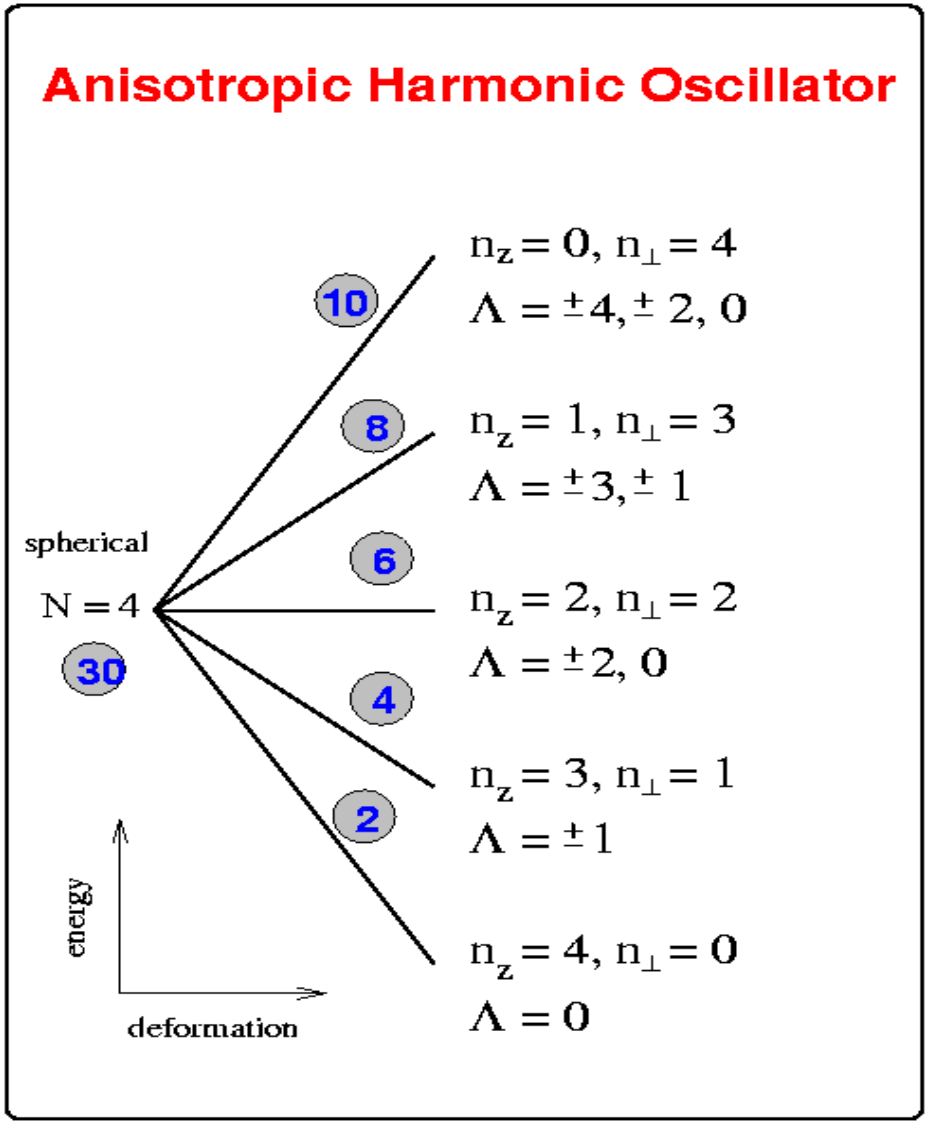
- Spin projections: $\Omega = \Lambda + \Sigma = \Lambda \pm \frac{1}{2}$

AHO Degeneracies

- Some of the degeneracies of the SHO are lifted
- Consider the N = 4 shell spherical oscillator shell which has degeneracy (N + 1)(N + 2) = 30 with ℓ = 4, 2, 0.
- The onset of deformation causes these levels to split into (N + 1) levels, each of degeneracy $2(n_\perp + 1)$:

n_z	$n_\perp$	Occupation
4	0	2
3	1	4
2	2	6
1	3	8
0	4	10

Levels of the AHO



- The splitting of the N = 4 oscillator shell is shown here when deformation is introduced
- Note that levels with large n_z (and hence small $n_\perp$) are favoured

Nilsson Model

- Nilsson added terms proportional to ℓ^2 and $\underline{\ell}.\underline{s}$ similar to the spherical case

- The resulting Modified Harmonic Oscillator (MHO) or Nilsson potential may be written as:

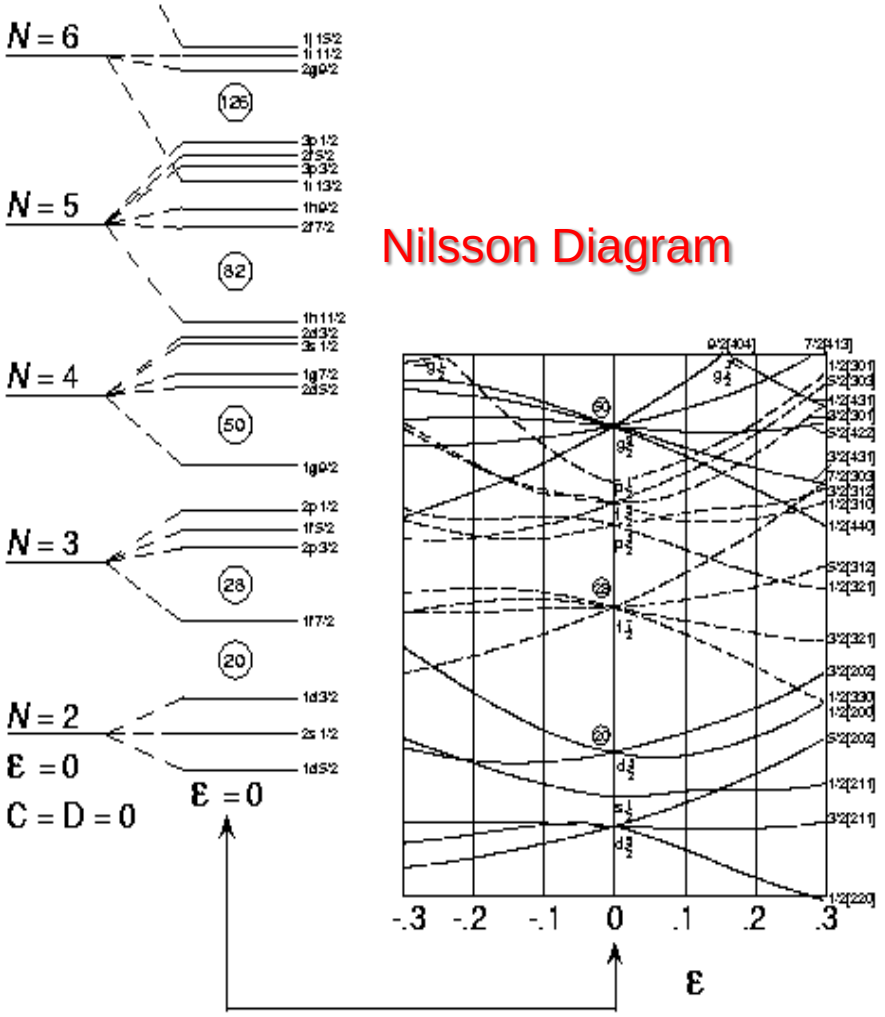
$$V_{MHO} = V_{osc} - \kappa\hbar\omega_0[2\underline{\ell}_t.\underline{s} + \mu(\ell_t^2 - \langle\ell_t^2\rangle_N]$$

where κ and μ are adjustable parameters. They are different for each major oscillator shell

- The $\underline{\ell}_t.\underline{s}$ term imitates the nuclear spin-orbit interaction in the stretched coordinate system

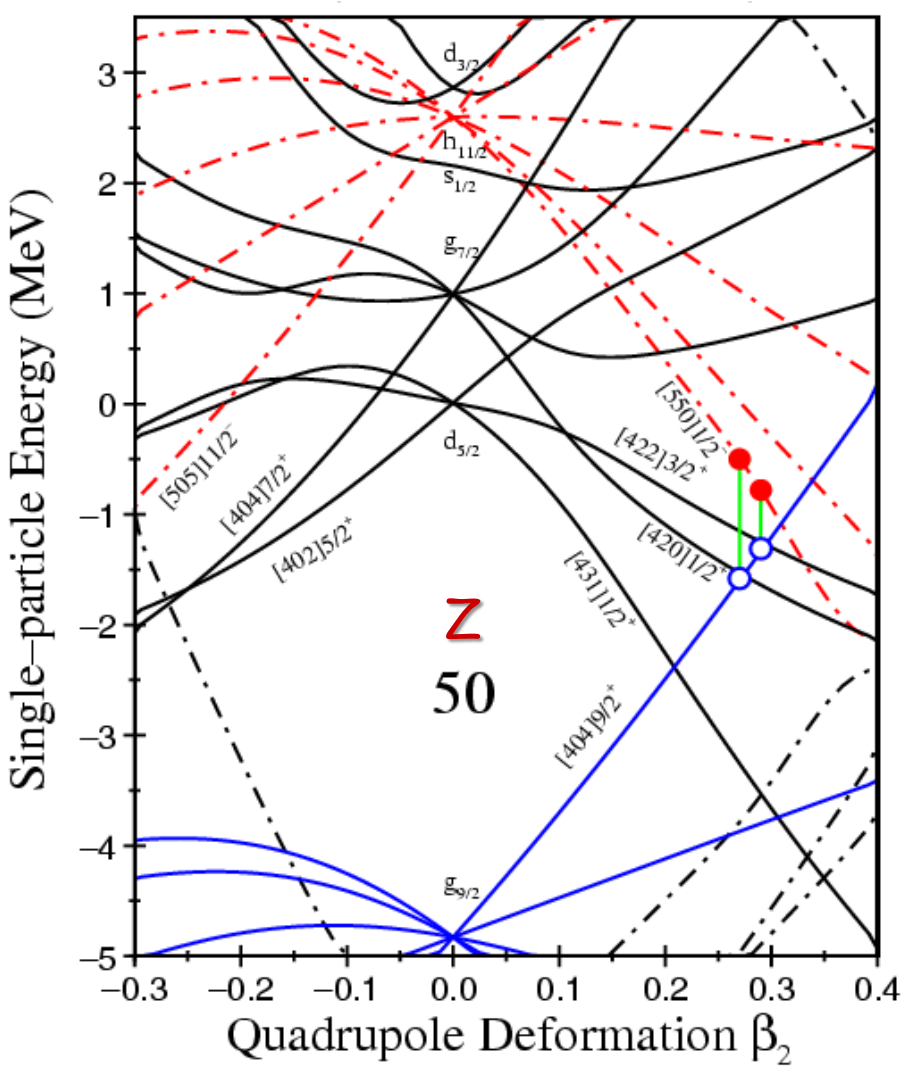
- The ℓ_t^2 term deepens the effective potential for particles near the nuclear surface

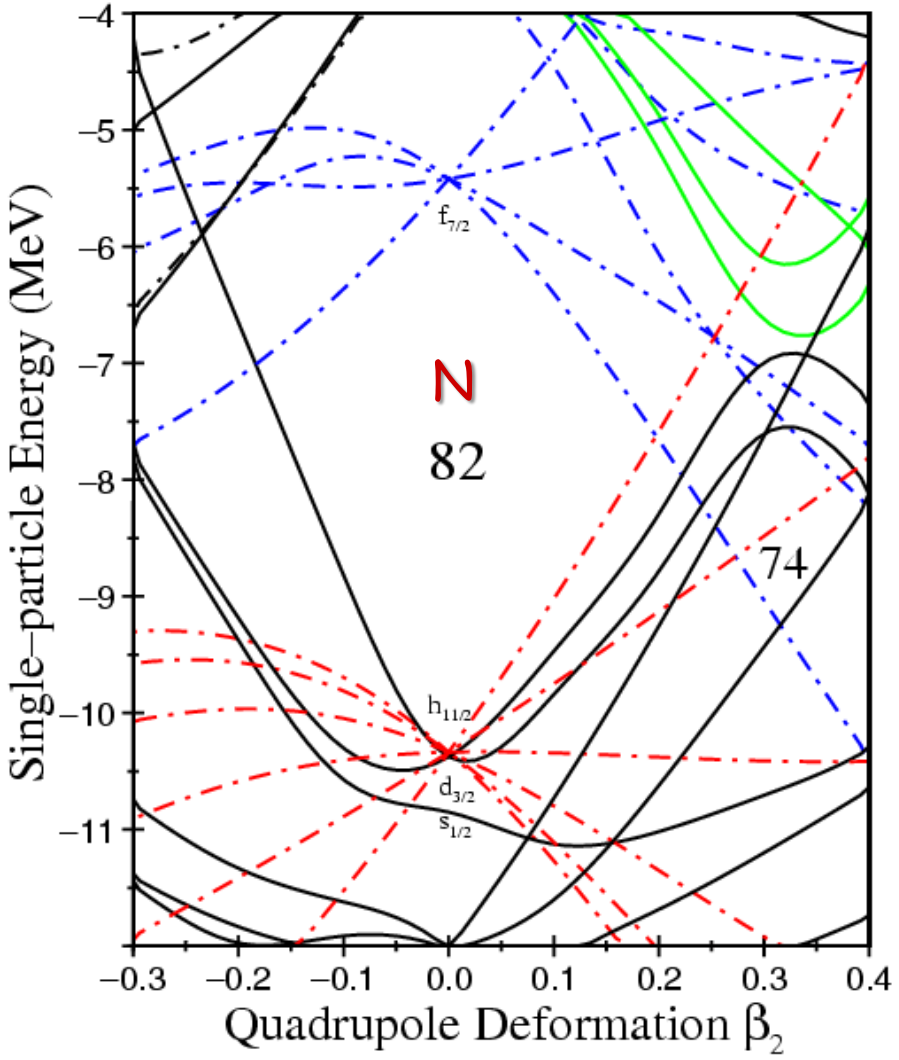
Remaining Degeneracies



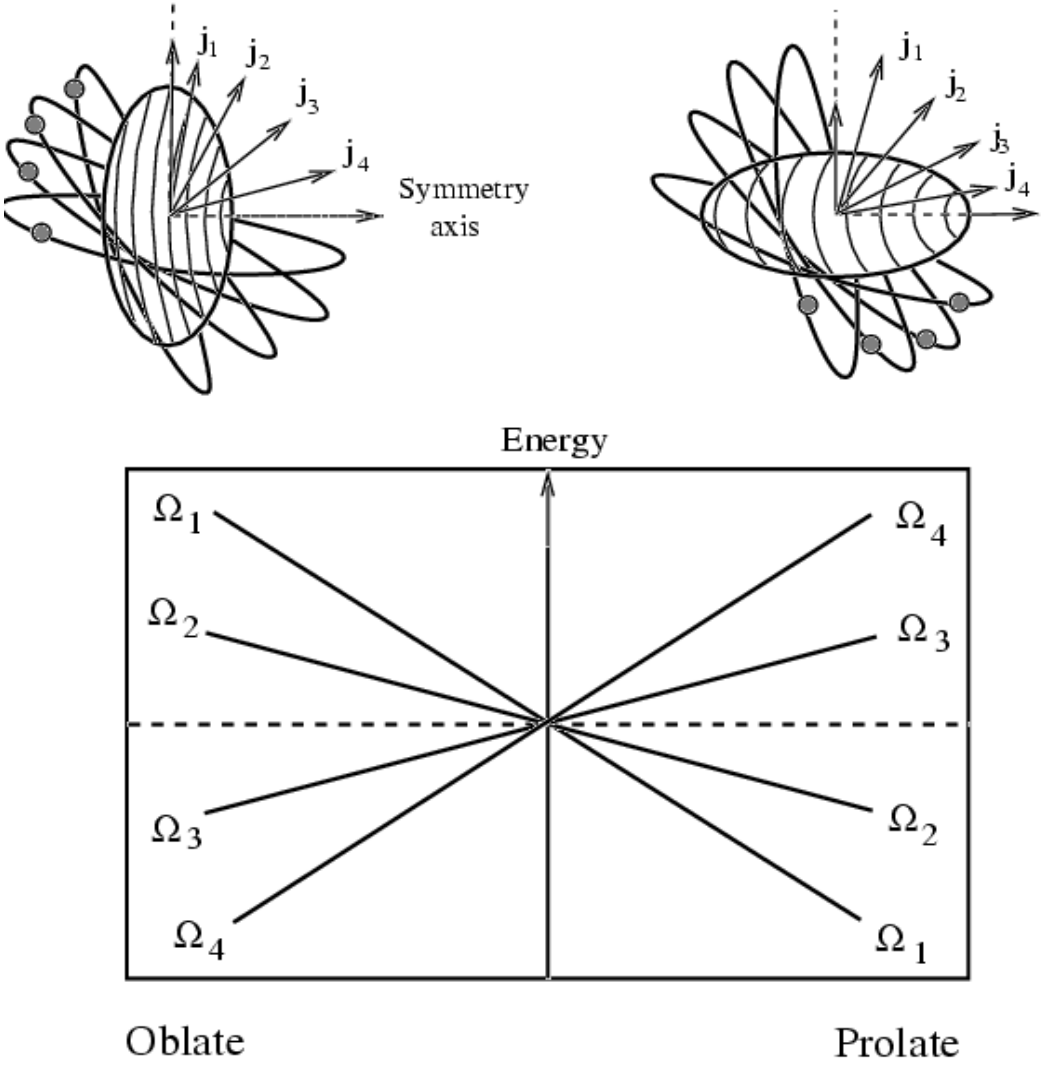
- The $\underline{\ell}_\perp.\underline{s}$ and $\ell_\perp^2$ terms lift the $2(n_\perp + 1)$ degeneracy of the $N = n_z + n_\perp$ states
- States with different Ω now have different energy
- Each $\Omega^\pi [N\ n_z\ \Lambda]$ state is only <u>twofold</u> degenerate, corresponding to particles with $\pm\Omega$

Nilsson Single-Particle Diagrams



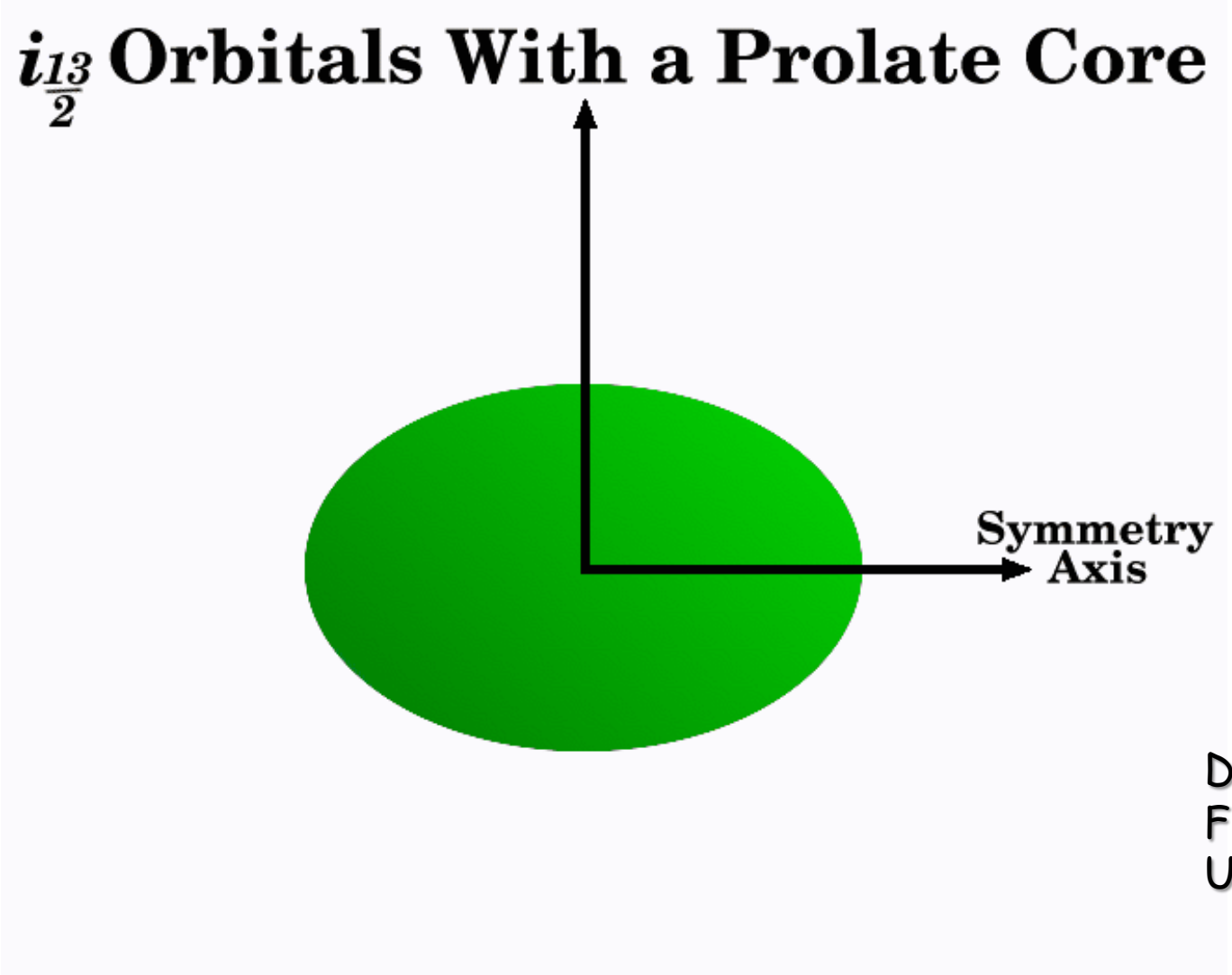


Splitting of Ω States



- Low Ω states favour prolate shapes
- High Ω states favour oblate shapes
- Note that each Ω state is now only twofold degenerate (±Ω)

Splitting of Ω States



David Campbell
Florida State University

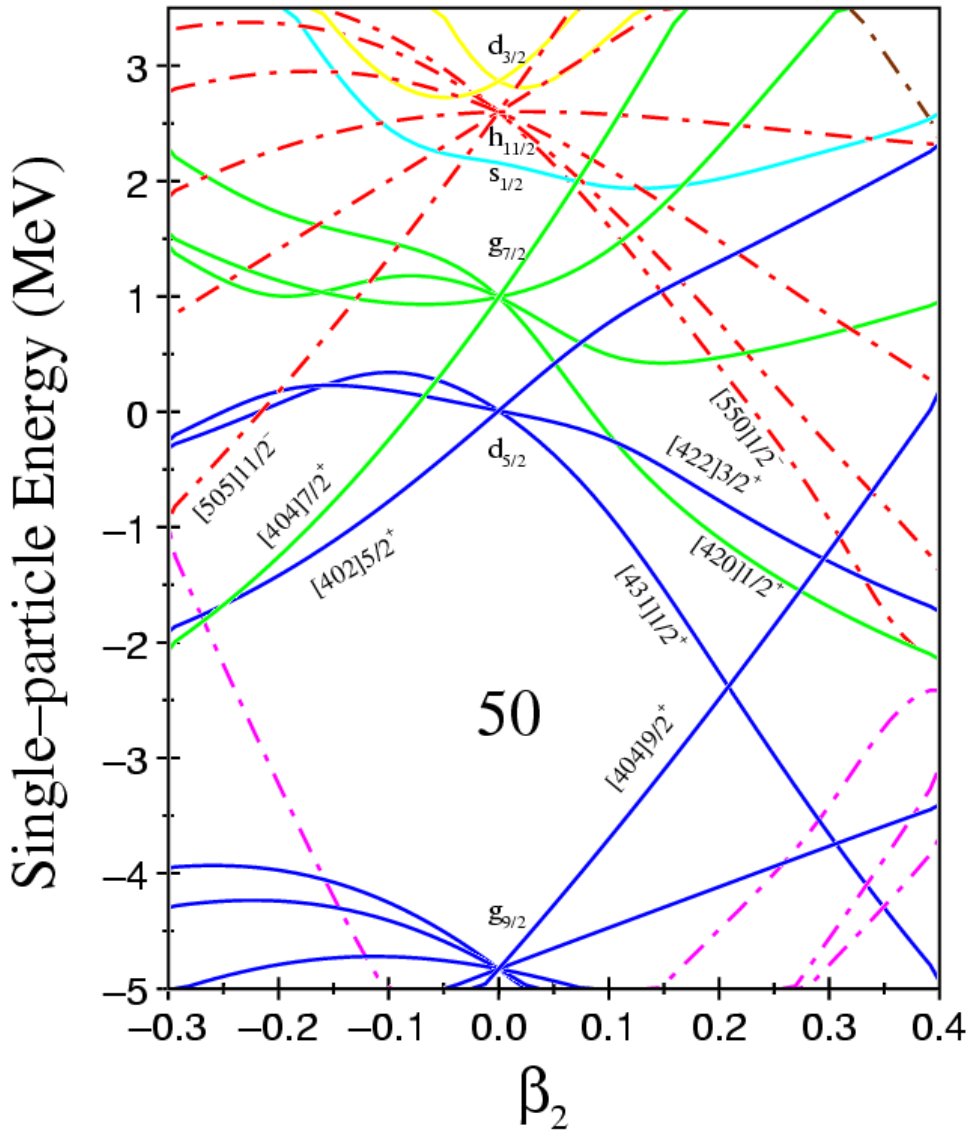
Asymptotic Quantum Numbers

- Because of the additional $\underline{\ell}.\underline{s}$ and ℓ^2 terms the physical quantities labelled by n_z and Λ are not constants of the motion, but only approximately so

- These quantum numbers are called asymptotic as they only come good as $\varepsilon_2 \rightarrow \infty$

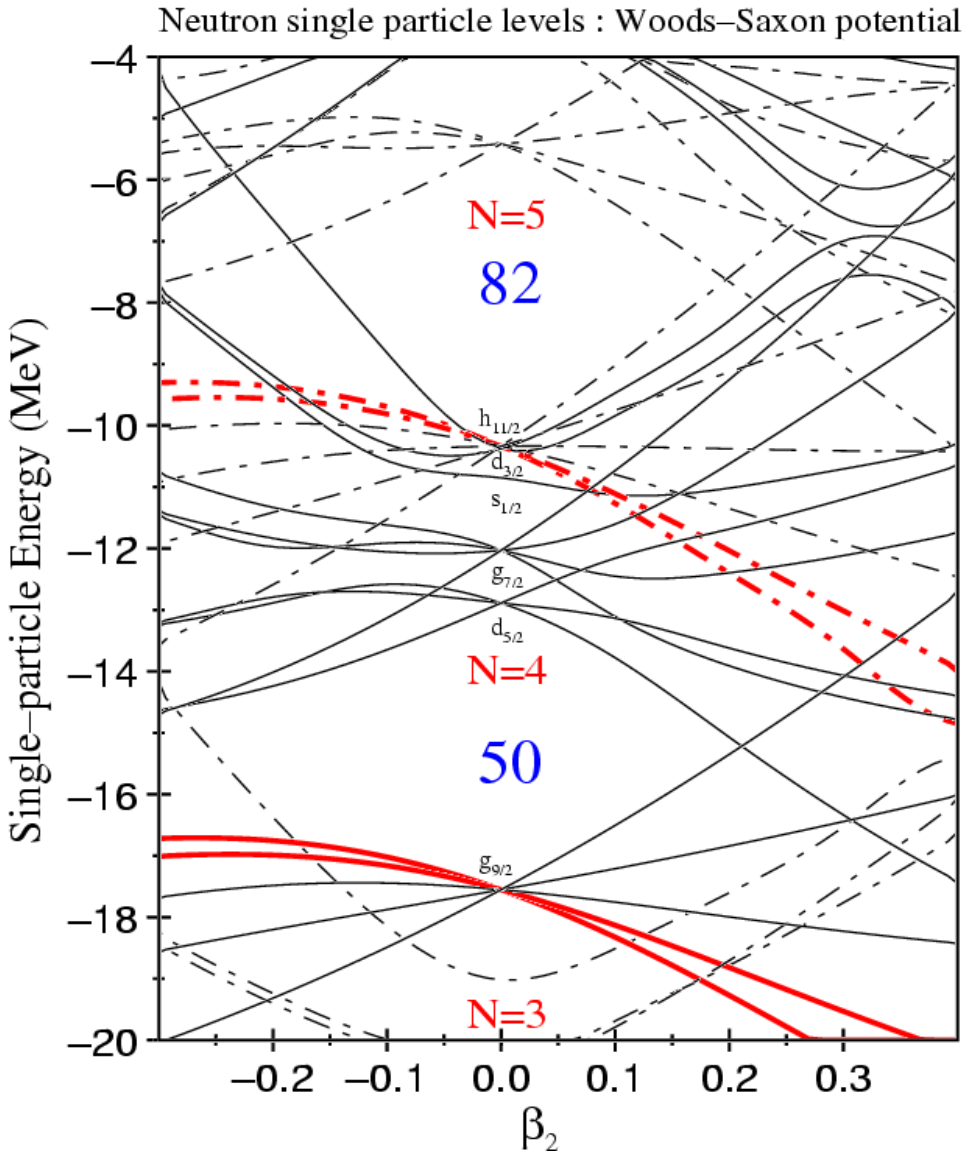
- However, the quantum numbers N, Ω and π are always good labels provided that:
  1. the nucleus is not rotating and
  2. there are no residual interactions

Proton Nilsson Diagram



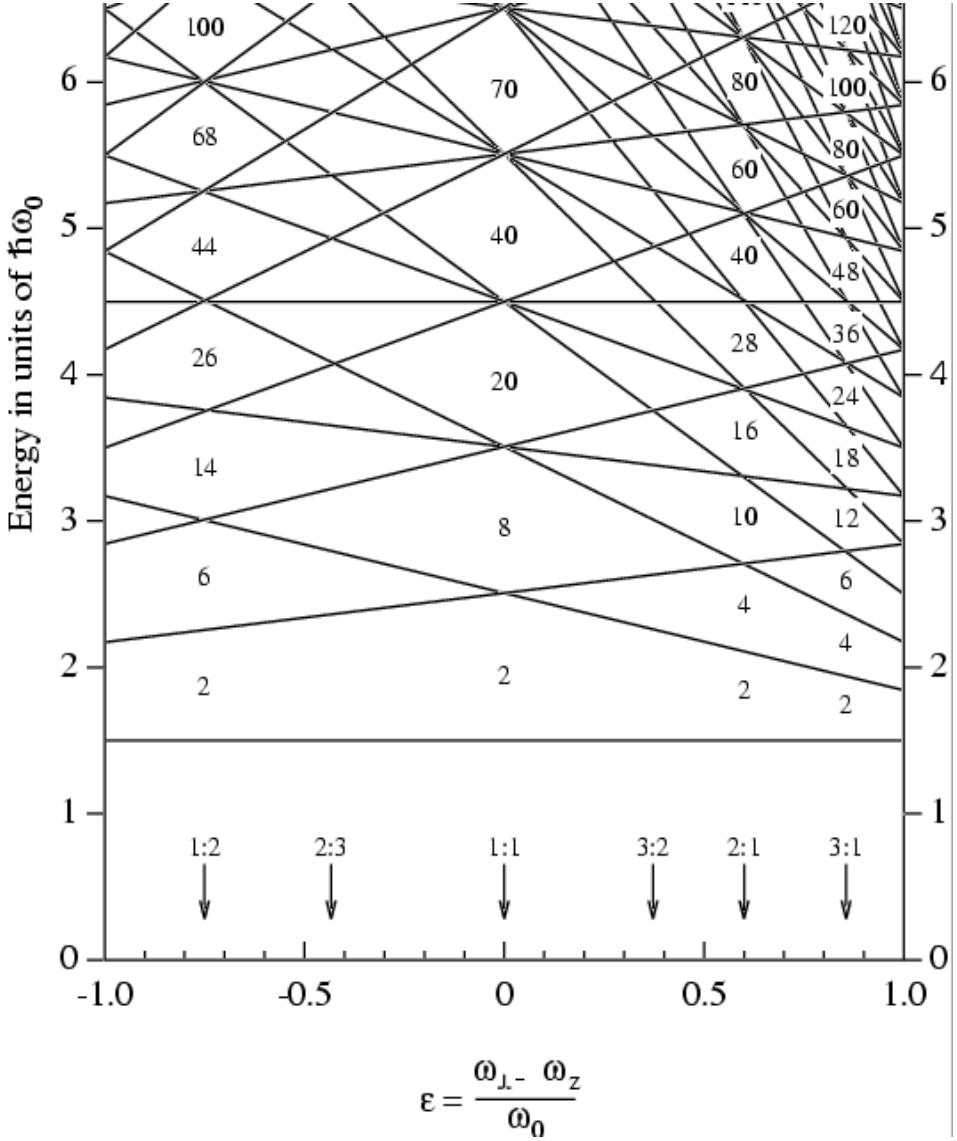
- A '<u>Nilsson Diagram</u>' shows nuclear energy levels as a function of a quadrupole deformation parameter (β_2, ε_2 or δ)
- In this diagram, the large spherical shell gap at Z = 50 is rapidly diminished by the onset of deformation for both prolate ($\beta_2 > 0$) and oblate ($\beta_2 < 0$) shapes

Intruder Orbitals



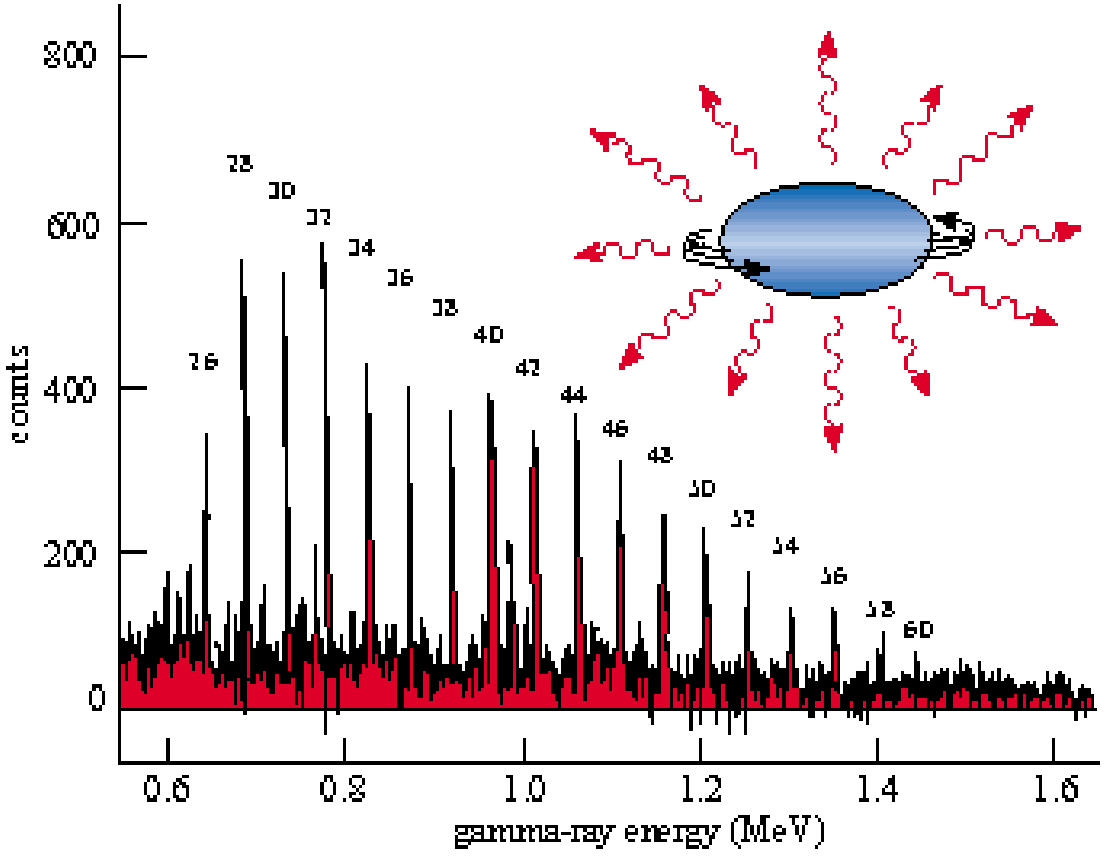
- The slope of Nilsson levels is related to the single-particle matrix element of the quadrupole operator: $dE/d\beta = -\langle k|r^2Y_{20}|k\rangle$
- Unnatural-parity low Ω prolate orbitals may 'intrude' down into a lower shell at large deformation
- This is the origin of superdeformation

Large Deformations



- Deformed shell gaps (new 'magic numbers') emerge when the ratio of the major and minor nuclear axes are equal to the ratio of small integers
- A superdeformed shape has a major to minor axis ratio of 2:1
- A hyperdeformed shape has a major to minor axis ratio of 3:1

Superdeformed ^{152}Dy



Original SD γ-ray spectrum from 1986 (Daresbury)

- The SD band in ^{152}Dy is a very regular structure with equally spaced gamma-ray transitions
- The spacing is relatively small, i.e. the band has a large moment of inertia (close to the rigid body value)

Superdeformed Axis Ratios

- The moment of inertia of a rigid sphere is:

  $\mathfrak{I}_{rig} = (A^{5/3}/72)\ \ \hbar^2\ MeV^{-1}$

- The moment of inertia of a prolate ellipsoid undergoing rigid rotation is:

  $\mathfrak{I}_{rig} = (A^{5/3}/72)\ (1 + x^2) / 2x^{2/3}\ \ \hbar^2\ MeV^{-1}$

  where x is the ratio of major to minor axes

- The moment of inertia is not always a good indicator of nuclear deformation (e.g. pairing)

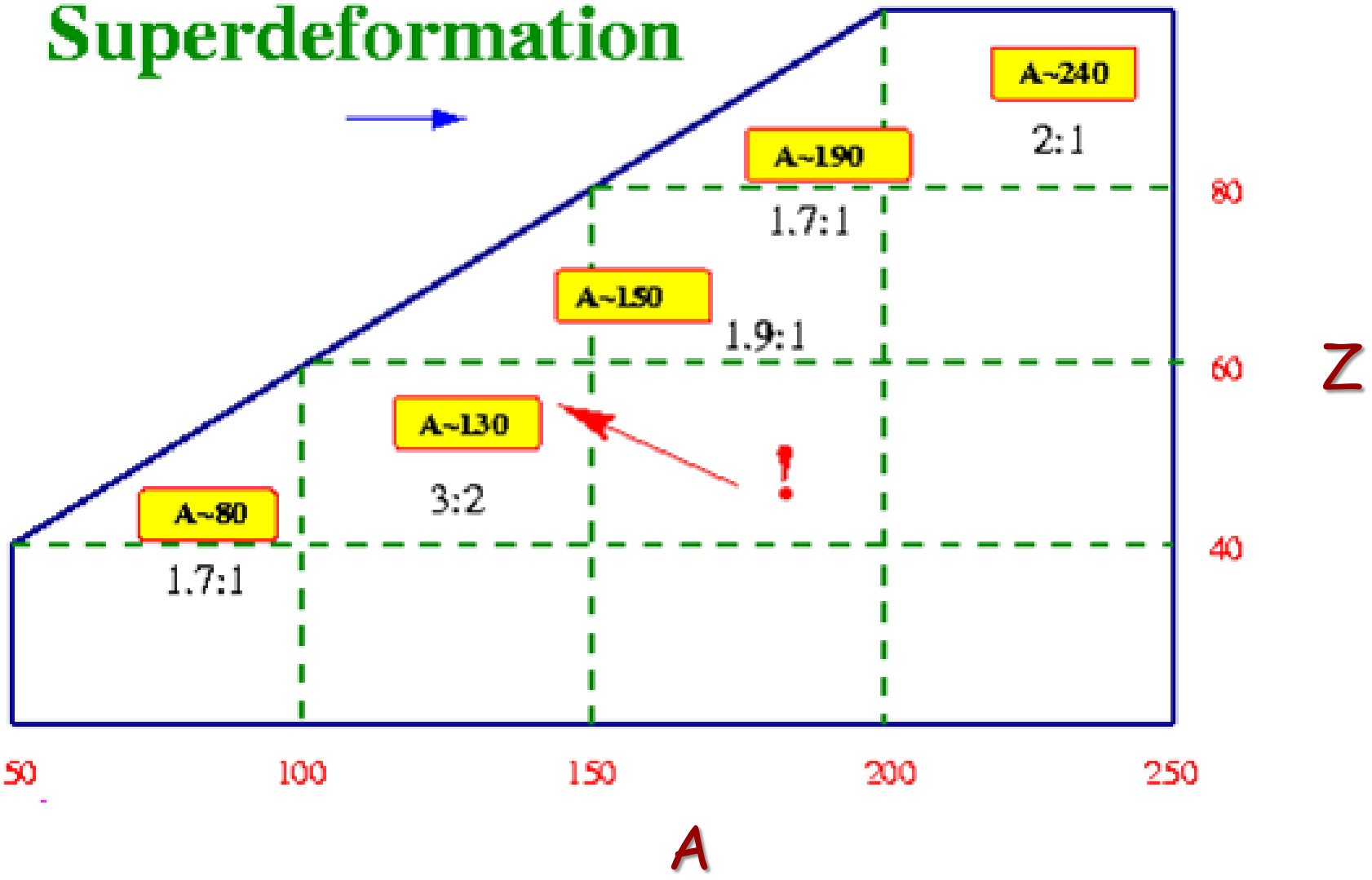
- The quadrupole moment (charge distribution) is a better indicator:

  $Q_0 = (2/5)\ Z\ R^2\ (x^2 - 1) / x^{2/3}\ \ eb$

SD Systematics

Nucleus	Q_0 (eb)	Axis Ratio
^{36}Ar	1.18	1.55
^{60}Zn	2.75	1.54
^{82}Sr	3.54	1.47
^{91}Tc	8.1	1.85
^{108}Cd	>9.5	>1.8
^{132}Ce	7.4	1.45
^{152}Dy	17.5	1.85
^{192}Hg	17.7	1.61
^{236}U	32	1.84

SD Regions


Superdeformation
A~240
2:1
A~190
1.7:1
A~150
1.9:1
A~130
3:2
!
A~80
1.7:1
80
60
40
Z
50
100
150
200
250
A